PETANQUE RATINGS

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Abstract

This paper provides a detailed analysis of three player rating systems that can be used in the sport of petanque. The first is a simple system based on a players ranking position (1st, 2nd, 3rd, etc.) in tournaments that they have played. The second system is more mathematical, involving a least squares estimate of a rating derived from a set of linear equations connecting winning and losing player ratings with match score differences. The third, the Elo[1] rating system, uses an assumed statistical model of player performance to predict the outcome of a match and then adjusts the player's rating (winner and loser) according to the actual match result. Examples of each system are provided.

Keywords

Petanque, Ratings, Least Squares, Elo Ratings, Logistic function

Introduction

Petanque is a sport where players may engage in singles games (one player against another player), doubles games (a team of two players against another team of two players) or triples games (a team of three players against another team of three players). In doubles or triples, teams can be male, female or mixed gender. Petanque games are usually played on a hard gravel surface (the terrain) which could be divided into a number of lanes (or pistes) with approximate dimensions of $15\text{ m} \times 4\text{ m}$. A team (or a player in singles) scores points for every boule closer to the jack than their opponent's boules. Boules are hollow metallic balls, between 70 mm and 80 mm diameter and weighing between 650 grams and 800 grams. A jack is a 30 mm diameter coloured wooden or plastic ball weighing between 10 and 18 grams. In singles or doubles games each player has three boules; in triples each player has two boules.

Games are usually played to 13 points and the result of a game is either a win or a loss; a draw is not possible. A team's score (points-for) and their opponent's score (points-against) give rise to a points difference, or delta where delta = points-for – points-against. For example Anne and Bob play a doubles game against Charles and Doreen. Anne and Bob win with a score of 13 points and Charles and Doreen score 5 points. Anne and Bob's $\text{delta} = 13 - 5 = +8$ and Charles and Doreen's $\text{delta} = 5 - 13 = -8$.

Petanque tournaments are a defined set of games, often a series of qualifying games followed by a finals series for the top-ranked teams. Tournaments are usually triples or doubles events and could include separate divisions for men and women; or they could be open, where male, female or mixed gender teams could play each other; or mixed tournaments where teams must include male and female players. Singles tournaments are less common.

To rank teams in a tournament – say for determining which teams should progress from the qualifying rounds to a finals series – it is usual to award 1 point for a win, and teams with an equal number of points (wins) are then separated by a tie-break or count-back procedure consisting of a number of levels. A common tie-break method is to use a team's accumulated delta scores. Higher delta scores rank above lower delta scores. If teams are still tied (same wins and delta) then the tie-break is each team's accumulated points-for. Higher points-for rank above lower points-for. It is not unheard of for teams to be tied on wins, delta and points-for and in this rare event a coin toss might be used to separate them. This tie-break

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[1] Arpad Elo (1903 – 1992) the Hungarian-born US physics professor and chess-master who devised a system to rate chess players that was implemented by the United States Chess Federation (USFC) in 1960 and adopted by the World Chess Federation (FIDE) in 1970. Elo described his work in his book *The Rating of Chess Players, Past & Present,* published in 1978 and his system has been adapted to many sports.

method (delta, then points-for, then coin toss) is only one of several methods that can be employed by tournament organisers. [See Example 7: Dove Open Doubles 01-May-2016, for a description of an alternative ranking system known as the Buchholtz system.]

In many tournaments, the draw (or sequence of games) in the qualifying rounds is random and it is possible for a strong team to play a number of weaker teams, or for the strongest teams in the tournaments to meet each other in the early rounds, and perhaps limiting their chances of proceeding to a finals series. To prevent this, the tournament organiser may choose to use the Swiss System[2], where in the first round, teams could be seeded, with the strong teams in a premier group not meeting each other, and subsequent rounds drawn from teams with the same or similar number of wins at that stage of the tournament. The Swiss System is often referred to as Winners-play-Winners.

For seeding in the first round of a Swiss System tournament it would be useful if teams were rated according to the strength of the players and their team-rating – assumed to be an average of the player's ratings – used to rank the teams from highest to lowest. Player or team-ratings could also be used for other purposes, say for selection of representative teams or players.

In petanque, players come together and form teams for a particular tournament. At the next tournament, the same players may combine in different teams and this presents problems in rating individual players. Our approach is to rate team performance (singles tournaments have teams of one) and then assign those team ratings to the players in the teams.

In this paper we will be investigating three rating systems that can be used to rank teams or players:

(i) A system used by Victoria Petanque Clubs Inc. (VPCI is an organisation of petanque clubs in the State of Victoria, Australia). This system uses a simple formula, combined with weighting factors for the class of tournaments, to assign tournament-points t based on a player's final ranking in a tournament. Averages of tournament-points provide player ratings.

(ii) A system based on the theory of Least Squares and systems of equations that assume linear relationships between player ratings and delta in a sequence of games in a tournament.

(iii) An Elo type rating system that assumes an underlying bell-shaped probability distribution and the likelihood of higher rating players beating lower rated players. Players of equal rating are assumed to have a 50-50 chance of success.

The terms rating and ranking are often taken to mean the same thing but in this paper:

a *rating* is derived from a mathematical model (a *rating system*) that assigns a number that indicates a player's strength relative to another; and

a *ranking* is an ordered list (highest to lowest) of ratings.

For example; Anne, Bob, Charles and Doreen have ratings of 365, 723, 1078 and 254 respectively. A ranking of these four players would be:

1	Charles	1078
2	Bob	723
3	Anne	365
4	Doreen	254

Charles would be regarded as the strongest player (rank = 1) and Doreen the weakest (rank = 4).

[2] The Swiss System (also known as the Swiss Ladder System) allows participants in a tournament to play a limited number of rounds against opponents of similar strength. The system was introduced in 1895 by Dr. J. Muller in a chess tournament in Zurich, hence the name 'Swiss System'. The principles of the system are: [1] In every round, each player is paired with an opponent with an equal score (or as nearly equal as possible); [2] Two players are paired at most once; [3] After a predetermined number of rounds the players are ranked according to a set of criteria. The leading player wins; or the ranking is the basis of subsequent elimination series.

Nomenclature

In this paper we have adopted the following notation

Symbol	Meaning	Definition
a	location parameter of Logistic function	
B	base-points in VPCI Player Rating System	
$\mathbf{B}$	(n,u) coefficient matrix of observation equations in least squares	
b	shape parameter of Logistic function	
dr_A, dr_B	difference in player/team ratings for A and B	
δ	delta	$\delta = \text{points for} - \text{points against}$
$\mathbf{f}$	$(n,1)$ vector of numeric terms of observation equations in least squares	
K	K-factor in Elo Rating System	
$\mathbf{N}$	(u,u) coefficient matrix of normal equations	$\mathbf{N} = \mathbf{B}^T\mathbf{W}\mathbf{B}$
n	number of equations in least squares solution	
P_A, P_B	probability of players/teams A and B winning	$P_A = \dfrac{1}{1+10^{-(r_A - r_B)/b}}$
φ	sum of squares of weighted residuals in least squares	$\varphi = \mathbf{v}^T\mathbf{W}\mathbf{v}$
M	Margin of Victory multiplier in Elo Rating System	
R	rank	
r	player/team rating	
r_A, r_B, r_W, r_L	ratings of players/teams A and B, winner (W), loser (L)	
s_W, s_L	score of winner, loser	
T	number of tournaments	
t	tournaments-points in VPCI Player rating System	
$\mathbf{t}$	$(u,1)$ vector of numeric terms of normal equations in least squares	$\mathbf{t} = \mathbf{B}^T\mathbf{W}\mathbf{f}$
u	number of unknowns in least squares solution	
v, $\mathbf{v}$	residual, $(n,1)$ vector of residuals in least squares	
w	weight of observation in least squares	
W_A, W_B	win/loss values for A and B in Elo Rating System	$W_A = \begin{Bmatrix} 1 \text{ if } A \text{ wins} \\ 0 \text{ if } A \text{ loses} \end{Bmatrix}, W_B = \begin{Bmatrix} 1 \text{ if } B \text{ wins} \\ 0 \text{ if } B \text{ loses} \end{Bmatrix}$
$\mathbf{W}$	(n,n) weight matrix in least squares	
x	variable	
$\mathbf{x}$	$(u,1)$ vector of unknowns in least squares solution	$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$

VPCI Player Rating System

In January 2015, the Committee of Victoria Petanque Clubs Inc. (VPCI), the Victorian State League of Petanque Federation Australia (PFA), adopted a Player Rating System and a Ranking of petanque players in Victoria. This Player Rating System was initially suggested by the then Secretary Peter Wells (Wells 2016) and modified in 2017 (Wells 2017) and again in 2018 (Wells 2018). The VPCI Player Rating System gives a rating of player performance based on a player's final ranking in tournaments. If a player is a member of a team then each member of the team receives the same rating points for that particular tournament. The VPCI Player Ratings and a Ranking of players is available on *Mypetanque*[3] (https://www.mypetanque.com).

The current VPCI Player Rating System (Wells 2018).

[The earlier versions of the system are explained in Appendix A.]

In this latest variation of the system, a player's rating r, which will be an integer value, is calculated from the following 2-step sequence.

[1] Using tournament results submitted to *Mypetanque,* a simple formula is used to assign tournament-points t to a player using base-points B according to the class of the tournament and the player's ranking R in a particular tournament

$$t = \frac{3\,B}{R+2} \tag{1}$$

Base-points B are fixed according to the tournament classes

Social/Regional Tournaments	$B = 500$
Club Hosted Tournaments	$B = 1000$
State Championships	$B = 1500$
PFA/National/International	$B = 2000$

The player's ranking R is: $R = 1$ for 1st place, $R = 2$ for 2nd place and so on.

If t not an integer, it is then rounded to the nearest integer.

[2] A player's rating r is then one of two values depending on the number of tournaments T they have played in the previous 12-month period:

(i) $T \leq 10$ (Less than or equal to 10 tournaments). Rating r is the sum of tournament-points t divided by 10*;* or

(ii) $T > 10$ (More than 10 tournaments). Rating r is the average of the tournament-points t of their 10 best results.

[3] *Mypetanque* is a website developed and maintained by Peter Wells and offered freely to the Australian petanque community. Using this website, petanque players may register for tournaments, view and download tournament details (Flyers) and view registered teams. Tournament results can be submitted and displayed and player ratings from a ranked list of male and female players are available for view and/or download. *Mypetanque* has access to PFA's national database of licensed players and also provides team data with rating information suitable for use in the SPORT software by Ottmar Kraemer-Fuhrmann (http://www.sport-software.de).

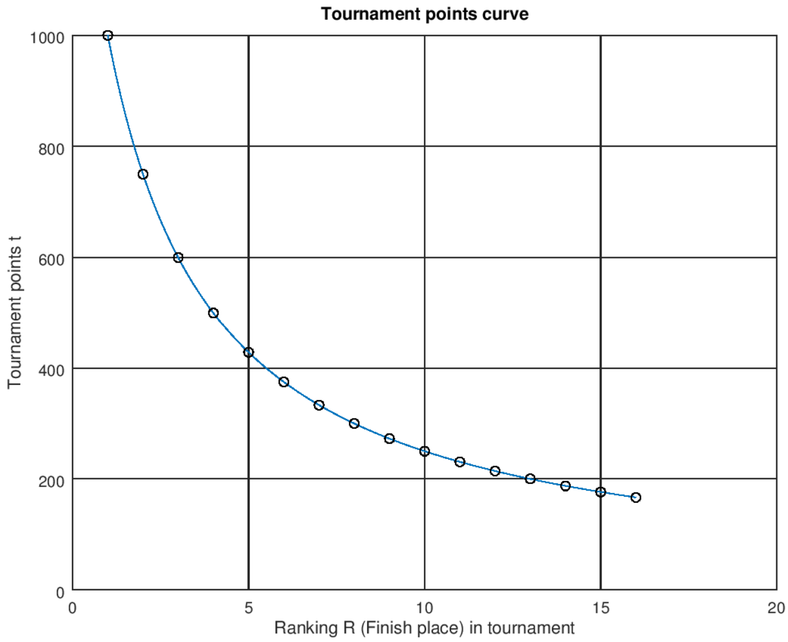

Figure 1. Tournament points curve $t = \dfrac{3\,B}{R+2}$ for a tournament with base-points $B = 1000$.
The small circles indicate the tournament-points for Ranks 1 to 16

Rank	Rating points t		**Rank**	Rating points t		**Rank**	Rating points t	
	Calculated	Rounded		Calculated	Rounded		Calculated	Rounded
1	1000	1000	**17**	157.895	158	**33**	85.714	86
2	750	750	**18**	150	150	**34**	83.333	83
3	600	600	**19**	142.857	143	**35**	81.081	81
4	500	500	**20**	136.364	136	**36**	78.947	79
5	428.571	429	**21**	130.435	130	**37**	76.923	77
6	375	375	**22**	125	125	**38**	75	75
7	333.333	333	**23**	120	120	**39**	73.171	73
8	300	300	**24**	115.385	115	**40**	71.429	71
9	272.727	273	**25**	111.111	111	**41**	69.767	70
10	2500	250	**26**	107.143	107	**42**	68.182	68
11	230.769	231	**27**	103.448	103	**43**	66.667	67
12	214.286	214	**28**	100	100	**44**	65.217	65
13	200	200	**29**	96.774	97	**45**	63.830	64
14	187.500	187	**30**	93.750	94	**46**	62.500	62
15	176.471	176	**31**	90.909	91	**47**	61.224	61
16	166.667	167	**32**	88.235	88	**48**	60	60

Table 1. Tournament-points t for a tournament with Base points $B = 1000$

Example 1.

A Club-Hosted Tournament $\left(B = 1000\right)$ where the top-8 from the qualifying rounds play a single elimination final series (Principale) and the remaining teams play a 3-round series (Consolante).

In the Principale the player finishing 1st $\left(R = 1\right)$ receives 1000 tournament-points and players finishing 2nd, 3rd and 4th receive 750, 600 and 500 tournament-points respectively according to $t = \dfrac{3B}{R+2}$.

$$
\begin{aligned}
R &= 1 \qquad t = \frac{3 \times 1000}{1+2} = \frac{3000}{3} = 1000 \\
R &= 2 \qquad t = \frac{3 \times 1000}{2+2} = \frac{3000}{4} = 750 \\
R &= 3 \qquad t = \frac{3 \times 1000}{3+2} = \frac{3000}{5} = 600 \\
R &= 4 \qquad t = \frac{3 \times 1000}{4+2} = \frac{3000}{6} = 500
\end{aligned}
$$

The losing quarter-finalists in the Principale are ranked as equal 5th $\left(R=5\right)$ and receive 429 tournament-points each.

$$
R = 5 \qquad t = \frac{3 \times 1000}{5+2} = \frac{3000}{7} = 428.571 \rightarrow 429
$$

In the Consolante, players finishing 1st, 2nd and 3rd receive 273, 250 and 231 tournament-points respectively and are ranked 9, 10 and 11 $\left(R = 9{,}10{,}11\right)$.

$$
\begin{aligned}
R &= 9 \qquad t = \frac{3 \times 1000}{9+2} = \frac{3000}{11} = 272.727 \rightarrow 273 \\
R &= 10 \qquad t = \frac{3 \times 1000}{10+2} = \frac{3000}{12} = 250 \\
R &= 11 \qquad t = \frac{3 \times 1000}{11+2} = \frac{3000}{13} = 230.769 \rightarrow 231
\end{aligned}
$$

Example 2.

A Club-Hosted Tournament $\left(B = 1000\right)$ where the top-8 from the qualifying rounds play a single elimination final series (Principale) and the remaining teams play a 3-round series (Consolante). The losing quarter-finalists in the Principale play a single elimination final series (Complémentaire).

In the Principale the player finishing 1st $\left(R = 1\right)$ receives 1000 tournament-points and players finishing 2nd, 3rd and 4th receive 750, 600 and 500 tournament-points respectively.

In the Complémentaire the player finishing 1st $\left(R = 5\right)$ receives 429 tournament-points and the players finishing 2nd, 3rd and 4th receive 375, 333 and 300 tournament-points respectively.

In the Consolante, the player finishing 1st $\left(R = 9\right)$ receives 273 tournament-points and the players finishing 2nd and 3rd receive 250 and 231 tournament-points respectively.

Example 3

Anne has played in $T = 4$ tournaments and her tournament-points t are shown in Table 2

Tournaments				Rank R and tournament-points t	
	Type	Class	Result	R	t
1	Doubles	Club ($B = 1000$)	equal 5th in Principale	5	429
2	Triples	Club ($B = 1000$)	2nd in Complémentaire	6	375
3	Triples	Regional (B =500)	3rd in Consolante	11	115
4	Doubles	State ($B = 2000$)	10th overall	10	500

Table 2. Tournament-points t for Anne's four tournaments

Since Anne has played in less than 10 tournaments ($T \leq 10$) her rating r will be the sum of the tournament-points divided by 10, or

$$r = \frac{\sum t}{10} = \frac{429 + 375 + 115 + 500}{10} = \frac{1419}{10} = 141.9 \rightarrow 142$$

Example 4

Charles has played in 13 tournaments ($T = 13$) in the past 12 months and his tournament-points are the set

$$t = \left\{ \begin{matrix} 429 & 167 & 231 & 600 & 600 \\ 750 & 900 & 429 & 500 & 1000 \\ 1000 & 250 & 333 & & \end{matrix} \right\}$$

Since Charles has played in more than 10 tournaments ($T > 10$) his rating will be the average of the 10 best tournament results, and since the average is not an integer, it is rounded down.

$$r = \frac{1000 + 1000 + 900 + 750 + 600 + 600 + 500 + 429 + 429 + 333}{10} = \frac{6541}{10} = 654.1 \rightarrow 654$$

The key points of the VPCI Rating System

- The system is simple and easy to understand.
- The formula for calculating tournament points t and ratings r are simple.
- All ratings (and tournament points) are positive integer values (whole numbers).
- The rating system is positively biased towards players with 10 or more competition results

As of 5th October 2018 there were 755 licensed petanque players in Victoria – 351 Female and 404 Male. There were 307 players (114 F, 193 M) who had a rating and 98 players (36 F, 62 M) with a rating $r \geq 200$. The top ranked female and male players had ratings of 1016 and 1263 respectively from 18 and 21 tournament results respectively in the preceding 12 months.

Player Ratings using Least Squares

Least Squares is a mathematical estimation process used to calculate the best estimate of unknown quantities from a set of n equations in u unknowns when $n > u$. It is commonly used to determine the line of best fit through a number of data points and this application is known as linear regression.

Least Squares can be used to determine player ratings from tournament results on the assumption that differences in game scores between winning and losing players is directly proportional to the difference in the ratings of the two players (Massey 1997, Langville & Meyer 2012). Equations linking game scores and player ratings (unknown quantities) will be greater than the number of player ratings and the *least squares principle* can be employed to determine the best estimates of the player ratings.

Least Squares: Brief history and a simple example

The first published work on least squares was by the French mathematician A.M. Legendre in 1805 (Nouvelles Methodes pour la Determination des Orbites des Cometes). Legendre's work of viii + 80 pages contained an Appendix of 9 pages where he set out his method "Sur la Methode des moindres quarres" and gave a worked example. Sur la Methode des moindres quarres translates to On the method of least squares.

C.F. Gauss (1809) published "Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium" [Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections] in which he states his rule: "*... the most probable system of values of the quantities ... will be that in which the sum of the squares of the differences between the actually observed and computed values multiplied by numbers that measure the degree of precision, is a minimum*" and bases this on his law of facility of error $y = ce^{-h^2x^2}$. We now know this as the 'normal' law of error (normal distribution). Gauss gave examples of his method of least squares and stated that he had been using this method since 1795.

This claim of priority in the discovery of the method of least squares sparked an international debate (Plackett 1972, Stigler 1981) but modern treatments of the method usually acknowledge Gauss as the inventor. Also, it has been demonstrated that the method does not require observations having particular statistical distributions, merely that they be free of observational blunders and systematic errors. And modern treatments use matrix algebra to describe the estimation process.

Both Gauss and Legendre developed the method of least squares in conjunction with studies in orbital mechanics, particularly Gauss who used the method to help rediscover the minor planet Ceres from earlier limited observations. And the logical extension of Gauss' least squares method is embodied in the Kalman Filter[4], a least-squares estimation process used to derive position of bodies in motion from measurements made at different instants of time. The Kalman Filter was an integral part of the navigation system of the Apollo spacecraft and is one of the most useful applications in modern electro-mechanical systems. GPS navigation and your FitBit watch wouldn't work without least squares (and the Kalman Filter).

A simple example below will demonstrate the Least Squares method and some definitions are useful.

First, it is assumed that we wish to estimate the values of certain quantities from measurements and that the nature of measurement means that every measurement contains errors. These errors may be classified as blunders, systematic errors and random errors. Blunders can be avoided by careful measurement process and checking and systematic errors can be eliminated or corrected by a proper understanding and calibration of measurement equipment and a knowledge of the environment in which the measurement is made.

Second, if blunders and systematic errors are eliminated, then the remaining random errors can be allowed for by the application of small corrections known as *residuals.* Hence we write

$$\text{measurement} + \text{residual} = \text{best estimate}$$

[4] Developed by Dr R.E. Kalman in 1960. The Kalman Filter is a recursive least squares estimation process particularly suited to dynamic problems associated with navigation. It regularly appears in lists of the most useful algorithms of the 20th century.

where ‘best estimate’ is a modern expression of Gauss’ ‘most probable value’.

Lastly, *weights* and *precision*. Often, a measurement may be the mean of several measurements or measurements may be obtained from different types of equipment or measurement processes and they may be of varying precision. To allow for this in least squares estimation we may weight our measurements, where a *weight* is a numerical value that reflects the degree of confidence we have in the measurement. The greater the weight the more confident we are in the particular measurement. A weight is often defined to be inversely proportional to the *variance* of a measurement where variance is a statistical measure of *precision*. Precise measurements have a small variance.

Example 5

Consider the simple problem of determining the distances x and y between three points A,B,C on a straight line.

Figure 2

The measurements are: $l_1 = AB = 87.420$ m , $l_2 = AC = 235.263$ m , $l_3 = BC = 147.865$ m and the weights of these measurements are: $w_1 = 9$, $w_2 = 12$, $w_3 = 3$ respectively.

Since we have $n = 3$ measurements in $u = 2$ unknowns (the distances x and y) then least squares can be employed as follows.

Write observation equations for each measurement

$$\begin{aligned} l_1 + v_1 &= x \\ l_2 + v_2 &= x + y \\ l_3 + v_3 &= y \end{aligned} \tag{2}$$

where v_i, v_2, v_3 are residuals (small unknown corrections)

Now the *least squares principle* is that the best estimates of x and y are those that make the sum of the squares of the residuals, multiplied by their weights, a minimum. To achieve this, write the *least squares function* φ as

$$\begin{aligned} \varphi\left(x,y\right) &= w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 \\ &= w_1\left(x - l_1\right)^2 + w_2\left(x + y - l_2\right)^2 + w_3\left(y - l_3\right)^2 \end{aligned}$$

Now $\varphi\left(x,y\right)$ will be a minimum when the partial derivatives $\dfrac{\partial\varphi}{\partial x}, \dfrac{\partial\varphi}{\partial y}$ both equal zero, that is

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= 2w_1\left(x - l_1\right) + 2w_2\left(x + y - l_2\right) = 0 \\ \frac{\partial\varphi}{\partial x} &= 2w_2\left(x + y - l_2\right) + 2w_3\left(y - l_3\right) = 0 \end{aligned}$$

Cancelling the 2’s and re-arranging gives the $u = 2$ *normal equations*

$$\begin{aligned} \left(w_1 + w_2\right)x + w_2 y &= w_1 l_1 + w_2 l_2 \\ w_2 x + \left(w_2 + w_3\right)y &= w_2 l_2 + w_3 l_3 \end{aligned}$$

Using the values above, the normal equations as

$$\begin{aligned} 21x + 12y &= 3609.936 \\ 12x + 15y &= 3266.751 \end{aligned} \tag{3}$$

which can be solved to give the best estimates $x = 87.4154$ m, $y = 147.8511$ m and the residuals $v_1 = -0.0046$ m, $v_2 = 0.0035$ m, $v_3 = -0.0139$ m

Least Squares and Matrices

The least squares process can be simplified with the use of matrix algebra and the following sequence

1. Write the n observation equations in the matrix form

$$\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f} \tag{4}$$

where $\mathbf{v}$ is a an $(n,1)$ vector of residuals, $\mathbf{B}$ is an (n,u) matrix of coefficients, $\mathbf{x}$ is a $(u,1)$ vector of unknowns and $\mathbf{f}$ is an $(n,1)$ vector of numeric terms. In the example above, we would re-arrange equations (2) as

$$\begin{aligned} v_1 - x &= -l_1 \\ v_2 - x - y &= -l_2 \\ v_3 - y &= -l_3 \end{aligned} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_1 \\ -l_2 \\ -l_3 \end{bmatrix}$$

2. Form the u normal equations

$$\left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{W}\mathbf{f} \quad \text{or} \quad \mathbf{N}\mathbf{x} = \mathbf{t} \tag{5}$$

where $\mathbf{N} = \mathbf{B}^T\mathbf{W}\mathbf{B}$ is a (u,u) symmetric coefficient matrix and $\mathbf{t} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ is a $(u,1)$ vector of numeric terms. The superscript ^{T} denotes matrix transpose, and $\mathbf{W}$ is an (n,n) diagonal matrix where the main diagonal elements are the weights $w_1, w_2, w_3, \ldots, w_n$. In the example above

$$\mathbf{N} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \left(w_1 + w_2\right) & w_2 \\ w_2 & \left(w_2 + w_3\right) \end{bmatrix} = \begin{bmatrix} 21 & 12 \\ 12 & 15 \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} -l_1 \\ -l_2 \\ -l_3 \end{bmatrix} = \begin{bmatrix} w_1 l_1 + w_2 l_2 \\ w_2 l_2 + w_3 l_3 \end{bmatrix} = \begin{bmatrix} 3609.936 \\ 3266.751 \end{bmatrix}$$

3. Solve the normal equations to obtain the $(u,1)$ vector of unknowns $\mathbf{x}$ using

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \tag{6}$$

where the superscript $^{-1}$ denotes matrix inverse defined as $\mathbf{N}\mathbf{N}^{-1} = \mathbf{I}$ and $\mathbf{I}$ is the Identity matrix.

4. Determine the residuals using (4) re-arranged as

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\mathbf{x} \tag{7}$$

The normal equations (5) are obtained by writing the least squares principle in matrix form as (Mikhail 1976)

$$\varphi = \mathbf{v}^T\mathbf{W}\mathbf{v} \;\Rightarrow\; \text{minimum} \tag{8}$$

$\mathbf{v}^T\mathbf{W}\mathbf{v}$ = the sum of the squares of residuals multiplied by the weight coefficients.

Using (4) and the rules of matrix transpose $\left(\left(\mathbf{ABC}\ldots\right)^T = \ldots\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T\right)$ and the fact that $\mathbf{W}$ is a symmetric matrix and $\mathbf{W}^T = \mathbf{W}$ we may write

$$\begin{aligned}\varphi &= \left(\mathbf{f}-\mathbf{Bx}\right)^T \mathbf{W}\left(\mathbf{f}-\mathbf{Bx}\right)\\ &= \left(\mathbf{f}^T-\mathbf{x}^T\mathbf{B}^T\right)\left(\mathbf{Wf}-\mathbf{WBx}\right)\\ &= \mathbf{f}^T\mathbf{Wf}-\mathbf{f}^T\mathbf{WBx}-\mathbf{x}^T\mathbf{B}^T\mathbf{Wf}+\mathbf{x}^T\mathbf{B}^T\mathbf{WBx}\\ &= \mathbf{f}^T\mathbf{Wf}-2\mathbf{f}^T\mathbf{WBx}+\mathbf{x}^T\mathbf{B}^T\mathbf{WBx}\end{aligned}$$

The minimum value of φ is obtained by partial differentiation with respect to the vector $\mathbf{x}$ and equating the result to a vector of zeros

$$\frac{\partial\varphi}{\partial\mathbf{x}} = -2\mathbf{f}^T\mathbf{WB}+2\mathbf{x}^T\mathbf{B}^T\mathbf{WB}=\mathbf{0}$$

Cancelling the 2's, re-arranging and transposing both sides gives the normal equations in matrix form

$$\left(\mathbf{B}^T\mathbf{WB}\right)\mathbf{x}=\mathbf{B}^T\mathbf{Wf} \quad \text{or} \quad \mathbf{Nx}=\mathbf{t}$$

Least Squares and Constraints

In some least squares applications it may be required that *constraints* be imposed, either to enable a solution of a singular system of equations or impose a set of mathematical restrictions on the unknowns $\mathbf{x}$. These constraint equations can be written in a matrix form as (Mikhail 1976)

$$\mathbf{Cx}=\mathbf{g} \tag{9}$$

where $\mathbf{C}$ is a (c,u) matrix of coefficients, $\mathbf{g}$ is a $(c,1)$ vector of numeric terms (constants) and c is the number of constraint equations.

The least squares principle with added constraints can be expressed as

$$\varphi=\mathbf{v}^T\mathbf{Wv}-2\mathbf{k}^T\left(\mathbf{Cx}-\mathbf{g}\right) \ \Rightarrow \ \text{minimum}$$

where $\mathbf{k}$ is a $(c,1)$ vector of *Lagrange multipliers* and noting that $\mathbf{Cx}-\mathbf{g}=\mathbf{0}$

This leads to the partitioned system of equations

$$\left[\begin{array}{c|c}-\mathbf{N} & \mathbf{C}^T\\ \hline \mathbf{C} & \mathbf{0}\end{array}\right]\left[\begin{array}{c}\mathbf{x}\\ \hline \mathbf{k}\end{array}\right]=\left[\begin{array}{c}-\mathbf{t}\\ \hline \mathbf{g}\end{array}\right] \tag{10}$$

and the solutions for $\mathbf{x}$ and $\mathbf{k}$ are obtained from

$$\left[\begin{array}{c}\mathbf{x}\\ \hline \mathbf{k}\end{array}\right]=\left[\begin{array}{c|c}-\mathbf{N} & \mathbf{C}^T\\ \hline \mathbf{C} & \mathbf{0}\end{array}\right]^{-1}\left[\begin{array}{c}-\mathbf{t}\\ \hline \mathbf{g}\end{array}\right] \tag{11}$$

We will use constraints and equations (10) and (11) to solve for player ratings.

Example 6

Suppose there are 4 teams in a round-robin tournament

Round 1	Round 2	Round 3
1(9) v **2**(13)	**1**(8) v **3**(13)	**1**(13) v **4**(2)
3(13) v **4**(6)	**4**(8) v **2**(13)	**2**(10) v **3**(13)

Table 3. Round-robin tournament for 4 teams
(game scores shown in parentheses beside team number)

We write an observation equation for each of the 6 games that has the general form (Massey 1997, Langville & Meyer 2012)

$$v+r_W-r_L=s_W-s_L \tag{12}$$

where r_W, r_L are ratings of the winning and losing teams respectively and s_W, s_L are scores of the winning and losing teams. v is a residual that reflects the fact that the scores of any game may not exactly accord with the ratings.

The $n = 6$ observation equations involving the $u = 4$ unknown ratings are

$$\begin{aligned}
v_1 + r_2 - r_1 &= 13 - 9 = 4 \\
v_2 + r_3 - r_4 &= 13 - 6 = 7 \\
v_3 + r_3 - r_1 &= 13 - 8 = 5 \\
v_4 + r_2 - r_4 &= 13 - 8 = 5 \\
v_5 + r_1 - r_4 &= 13 - 2 = 11 \\
v_6 + r_3 - r_2 &= 13 - 10 = 3
\end{aligned}$$

and they can be expressed in the matrix form $\mathbf{v} + \mathbf{Bx} = \mathbf{f}$ as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \\ 5 \\ 11 \\ 3 \end{bmatrix}$$

If we say all games have equal weight we can write $\mathbf{W} = \mathbf{I}$ (the Identity matrix) and the normal equation coefficient matrix $\mathbf{N}$ and the numeric terms $\mathbf{t}$ have the simplified form and values

$$\mathbf{N} = \mathbf{B}^T\mathbf{B} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \qquad \mathbf{t} = \mathbf{B}^T\mathbf{f} = \begin{bmatrix} 2 \\ 6 \\ 15 \\ -23 \end{bmatrix}$$

Note here that vector $\mathbf{t}$ is the accrued 'delta' for the 4 players where delta = points *for* – points *against*.

It turns out that the symmetric matrix $\mathbf{N}$ is rank deficient (its rank is 3) and $\mathbf{N}^{-1}$ is not defined and there can be no solution of the normal equations $\mathbf{Nx} = \mathbf{t}$. To overcome this problem a single constraint equation of the form $\mathbf{Cx} = \mathbf{g}$ can be added.

Suppose that this single constraint is that the sum of the ratings must equal 100

$$r_1 + r_2 + r_3 + r_4 = 100$$

and in the form $\mathbf{Cx} = \mathbf{g}$ we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 100 \end{bmatrix}$$

With the added constraint equation the system of equations for a solution of the ratings is given by (10)

$$\left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{g} \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right] \left[\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ \hline k_1 \end{array}\right] = \left[\begin{array}{c} -2 \\ -6 \\ -15 \\ 23 \\ \hline 100 \end{array}\right]$$

The solution of this system of equations is given by (11) from which we obtain the vector $\mathbf{x}$ (ratings) as

$$\mathbf{x} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 25.50 \\ 26.50 \\ 28.75 \\ 19.25 \end{bmatrix}$$

and the sum of the ratings = 100.

The vector of residuals (transposed) is

$$\mathbf{v}^T = \begin{bmatrix} 3.00 & -2.50 & 1.75 & -2.25 & 4.75 & 0.75 \end{bmatrix}$$

and these values reflect the difference between the ratings and the game scores. For example, the observation equation for the first match (1 v 2) is $v_1 + r_2 - r_1 = 13 - 9 = 4$ and the difference in ratings is $r_2 - r_1 = 26.5 - 25.5 = 1$ point. The actual match score difference was 4 points and the residual $v_1 = 3$ points. So the 'expected' result (according to the ratings) was a win for player 2 by 1 point and the actual score was a win by 4 points.

Example 7. Dove Open Doubles 01-May-2016

In this tournament there were 18 teams. There were 4 Qualifying rounds (Swiss System) and the top 9 teams were seeded and did not play each other in Round 1 of the Qualifying. After the Qualifying the teams were ranked 1–18 and teams ranked 1 to 8 went into a Principale and teams ranked 9 to 16 went into a Complémentaire. The remaining two teams took no further part in the tournament. The Principale and Complémentaire were single elimination finals series with play-off's for 3rd and 4th places. There were 36 matches in the Qualifying and 8 matches each in the Principale and Complémentaire making a total of 52 matches involving the 18 teams.

Tables 4 (Qualifying), 6 (Principale) and 7 (Complémentaire) show matches and game scores in the tournament and Tables 5 and 8 show ranking after Qualifying and final ranking in Principale and Complémentaire respectively.

Round 1			Round 2			Round 3			Round 4		
16(13)	v	**9**(4)	**2**(7)	v	**14**(11)	**3**(13)	v	**14**(8)	**3**(13)	v	**16**(8)
18(6)	v	**14**(10)	**7**(9)	v	**15**(13)	**15**(11)	v	**16**(12)	**18**(12)	v	**7**(13)
5(12)	v	**11**(8)	**8**(3)	v	**16**(13)	**17**(11)	v	**6**(12)	**14**(5)	v	**4**(10)
7(13)	v	**1**(6)	**5**(5)	v	**17**(13)	**5**(10)	v	**18**(13)	**15**(13)	v	**6**(11)
2(13)	v	**12**(0)	**3**(13)	v	**6**(3)	**4**(13)	v	**13**(12)	**1**(12)	v	**13**(9)
8(13)	v	**4**(8)	**4**(10)	v	**11**(6)	**8**(8)	v	**1**(9)	**5**(8)	v	**2**(13)
15(13)	v	**17**(1)	**9**(2)	v	**18**(13)	**2**(10)	v	**7**(13)	**17**(12)	v	**8**(11)
13(10)	v	**6**(13)	**13**(13)	v	**12**(2)	**11**(13)	v	**10**(4)	**11**(11)	v	**12**(8)
3(13)	v	**10**(7)	**10**(3)	v	**1**(13)	**9**(8)	v	**12**(9)	**10**(9)	v	**9**(11)

Table 4. Dove Open Doubles: Qualifying matches
(game scores shown in parentheses beside team number)

Qualifying Ranking							
Rank	Team	Score	BHN	fBHN	Games	Points	delta
1	**3**	4	7	40	4:0	52:26	+26
2	**15**	3	10	36	3:1	50:33	+17
3	**7**	3	10	29	3:1	48:41	+7
4	**16**	3	9	34	3:1	46:31	+15
5	**4**	3	6	36	3:1	41:36	+5
6	**1**	3	5	40	3:1	40:33	+7
7	**14**	2	11	27	2:2	34:36	−2
8	**6**	2	10	33	2:2	39:47	−8
9	**17**	2	7	39	2:2	37:41	−4
10	**2**	2	7	35	2:2	43:32	+11
11	**18**	2	7	35	2:2	44:35	+9
12	**11**	2	5	30	2:2	38:34	+4
13	**8**	1	11	27	1:3	35:42	−7
14	**13**	1	9	27	1:3	44:40	+4
15	**5**	1	8	26	1:3	35:47	−12
16	**9**	1	6	32	1:3	25:44	−19
17	**12**	1	6	27	1:3	19:45	−26
18	**10**	0	10	23	0:4	23:50	−27

Table 5. Dove Open Doubles: Ranking after Qualifying rounds
(BHN is Buchholtz Number[5], fBHN is Fine Buchholtz Number)

Quarter-Finals	Semi-Finals	Final	Playoff
3(11) v **6**(9)	**3**(13) v **16**(2)	**3**(13) v **14**(7)	**16**(6) v **7**(13)
16(11) v **4**(10)	**7**(8) v **14**(9)		
1(9) v **7**(11)			
14(11) v **15**(9)			

Table 6. Dove Open Doubles: Principale matches

Quarter-Finals	Semi-Finals	Final	Playoff
17(10) v **9**(8)	**17**(2) v **11**(13)	**11**(10) v **18**(13)	**17**(3) v **2**(13)
11(13) v **8**(4)	**18**(13) v **2**(1)		
13(2) v **18**(13)			
5(7) v **2**(13)			

Table 7. Dove Open Doubles: Complémentaire matches

Principale		Complémentaire	
Rank	Team	Rank	Team
1	**3**	1	**18**
2	**14**	2	**11**
3	**7**	3	**2**
4	**16**	4	**17**
= 5	**4** **6** **15** **1**	= 5	**8** **13** **9** **5**

Table 8. Dove Open Doubles: Final Ranking in Principale & Complémentaire

[5] The Buchholtz system is a ranking system, first used by Bruno Buchholtz in a Swiss System chess tournament in 1932. The principle of the system is that when two players have equal scores at the end of a defined number of rounds a tie break is required to determine the top ranked player. The scores of both player's opponents (in all rounds) are added giving each their Buchholtz Number (BHN). The player having the larger BHN is ranked higher on the assumption they have played against better performing players. The Fine Buchholtz Number (fBHN) is the sum of the opponents' Buchholtz Numbers and is used to break ties where player's BHN are equal. In the rare case that Score, BHN and fBHN are all equal then delta = points For – points Against is used as a tie break (see Teams 2 & 18 in Table 7).

Using the tournament match results and least squares we may determine the $u = 18$ (unknown) ratings $r_1, r_2, r_3, \ldots, r_{18}$ of the teams in a number of ways, three of which are:

1. from the Qualifying rounds only (36 matches)
2. from the Qualifying + Principale + Complémentaire (52 matches)
3. from the Qualifying + Principale + Complémentaire, but with differential weighting

Option 1: Ratings from Qualifying ($n = 36$ matches)

Following *Example 7* we may write an observation equation of the form of (12) for each of the $n = 36$ matches of the Qualifying rounds involving the $u = 18$ (unknown) ratings of the teams.

$$v + r_W - r_L = s_W - s_L$$

r_W, r_L are ratings of the winning and losing teams respectively and s_W, s_L are scores of the winning and losing teams. v is a residual that reflects the fact that the scores of any match may not exactly accord with the ratings.

The observation equations for the first round matches of the Qualifying are

$$\begin{aligned}
v_1 + r_{16} - r_9 &= 13 - 4 = 9 \\
v_2 + r_{14} - r_{18} &= 10 - 6 = 4 \\
v_3 + r_5 - r_{11} &= 12 - 8 = 4 \\
v_4 + r_7 - r_1 &= 13 - 6 = 7 \\
v_5 + r_2 - r_{12} &= 13 - 0 = 13 \\
v_6 + r_8 - r_4 &= 13 - 8 = 5 \\
v_7 + r_{15} - r_{17} &= 13 - 1 = 12 \\
v_8 + r_6 - r_{13} &= 13 - 10 = 3 \\
v_9 + r_3 - r_{10} &= 13 - 7 = 6
\end{aligned}$$

Writing these equations in the matrix form $\mathbf{v} + \mathbf{Bx} = \mathbf{f}$ (together with the observation equations for matches 19, 28, 35 and 36) gives the following structure

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ \vdots \\ \vdots \\ v_{19} \\ \vdots \\ \vdots \\ v_{28} \\ \vdots \\ v_{35} \\ v_{36} \end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \\ r_{15} \\ r_{16} \\ r_{17} \\ r_{18} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 4 \\ 7 \\ 13 \\ 5 \\ 12 \\ 3 \\ 6 \\ \vdots \\ \vdots \\ 5 \\ \vdots \\ \vdots \\ 5 \\ \vdots \\ 3 \\ 2 \end{bmatrix} \quad (13)$$

$\mathbf{v}$ is an $(n,1)$ vector of residuals and $\mathbf{B}$ is an (n,u) coefficient matrix with the n rows corresponding with the matches and u columns with the ratings. In each row of $\mathbf{B}$ there will be a 1 and a -1 in columns that correspond with the winning and losing teams respectively. $\mathbf{x}$ is a $(u,1)$ column vector of unknown ratings and $\mathbf{f}$ is an $(n,1)$ vector of numeric terms that are points differences for each match.

If each match of the Qualifying is regarded as having the same importance and each match is independent of other matches, then we may assign a weight $w = 1$ to each observation equation and assume a diagonal weight matrix $\mathbf{W} = \mathbf{I}$ of dimensions (n,n).

With $\mathbf{W} = \mathbf{I}$, the normal equations are $\left(\mathbf{B}^T\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{f}$ or $\mathbf{Nx} = \mathbf{t}$ and the solution for the ratings in $\mathbf{x}$ is given by $\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$, but the normal equation coefficient matrix $\mathbf{N}$ (shown below) is rank deficient (rank = 17) and the inverse is not defined.

$$\mathbf{N} = \begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4
\end{bmatrix}$$

The transpose of the vector of numeric terms $\mathbf{t}$ is

$$\mathbf{t}^T = \begin{bmatrix} 7 & 11 & 26 & 5 & -12 & -8 & 7 & -7 & -19 & -27 & 4 & -26 & 4 & -2 & 17 & 15 & -4 & 9 \end{bmatrix}$$

and these are the accumulated delta for each team (see Table 7 noting that the order is different).

To overcome the problem of rank deficiency in $\mathbf{N}$ a single constraint equation of the form $\mathbf{Cx} = \mathbf{g}$ can be added. Let this equation be, that the sum of the rankings equal zero, or

$$r_1 + r_2 + r_3 + \cdots + r_{17} + r_{18} = 0$$

where $\mathbf{C}$ is $(1,u)$ row vector of ones and $\mathbf{g}$ is a vector containing a single value equal to zero or

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{17} \\ r_{18} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

With the added constraint equation the system of equations for a solution of the ratings is given by (10) as

$$\left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{g} \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} -4 & 0 & \cdots & 0 & 1 \\ 0 & -4 & \cdots & 0 & 1 \\ \vdots & 1 & -4 & \vdots & \vdots \\ 0 & 0 & \cdots & -4 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right]\left[\begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_{18} \\ \hline k_1 \end{array}\right] = \left[\begin{array}{c} -7 \\ -11 \\ \vdots \\ -9 \\ \hline 0 \end{array}\right]$$

The solution of this system of equations is given by (11) from which we obtain the vector $\mathbf{x}$ (ratings) as

$$\mathbf{x} = \left\{\begin{array}{lll} r_1 = 0.820 & r_7 = 4.357 & r_{13} = -0.538 \\ r_2 = 1.250 & r_8 = 0.073 & r_{14} = 1.987 \\ r_3 = 6.816 & r_9 = -7.264 & r_{15} = 7.150 \\ r_4 = 0.676 & r_{10} = -7.612 & r_{16} = 5.444 \\ r_5 = -3.252 & r_{11} = -3.820 & r_{17} = 0.354 \\ r_6 = 1.445 & r_{12} = -9.093 & r_{18} = 1.207 \end{array}\right\} \tag{14}$$

The sum of the ratings in $\mathbf{x}$ is zero as per the constraint equation and the largest rating is team 15 with $r_{15} = 7.150$ and the lowest rating is team 12 with $r_{12} = -9.093$.

Suppose that we want the highest rating team to have a rating of 100, therefore 92.850 must be added to each value in the set (14) giving the ratings as

$$\text{ratings} = \left\{\begin{array}{lll} r_1 = 93.670 & r_7 = 97.207 & r_{13} = 92.312 \\ r_2 = 94.100 & r_8 = 92.923 & r_{14} = 94.837 \\ r_3 = 99.666 & r_9 = 85.586 & r_{15} = 100.000 \\ r_4 = 93.526 & r_{10} = 85.238 & r_{16} = 98.294 \\ r_5 = 89.598 & r_{11} = 89.030 & r_{17} = 93.204 \\ r_6 = 94.295 & r_{12} = 83.757 & r_{18} = 94.057 \end{array}\right\}$$

A Table of teams and least squares ratings from the Qualifying rounds is shown below ranked from largest to smallest ratings

Rank	Team	Rating	Rank	Team	Rating	Rank	Team	Rating
1	**15**	100	7	**2**	94.100	13	**13**	92.312
2	**3**	99.666	8	**18**	94.057	14	**5**	89.598
3	**16**	98.294	9	**1**	93.670	15	**11**	89.030
4	**7**	97.207	10	**4**	93.526	16	**9**	85.586
5	**14**	94.837	11	**17**	93.204	17	**10**	85.238
6	**6**	94.2950	12	**8**	92.923	18	**12**	83.757

Table 9. Ranking of Least Squares Ratings (from Qualifying rounds)

It is interesting to note that the top six ranked teams after the Qualifying rounds using the Buchholtz system (see Table 5) are 3, 15, 7, 16, 4 and 1. This order is quite different from the Least Squares ranking and is due to the fact that the Buchholtz system places much higher value on wins (1 point) and the wins of opponents (BHN) whereas the Least Squares ranking is dependent on wins and delta only.

Option 2: Ratings from Qualifying + Principale + Complémentaire ($n = 52$ matches)

We can use the observation equations for the Qualifying (36 matches) already shown in (13) and add in the observation equations for the Principale (8 matches) and Complémentaire (8 matches) giving a set of $n = 52$ equations in $u = 18$ unknown ratings in the matrix form $\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f}$. It should be noted that all matches (Qualifying, Principale, Complémentaire) are regarded as having the same importance and each match is independent of other matches. Hence each observation equation is assigned a weight $w = 1$ forming a diagonal weight matrix $\mathbf{W} = \mathbf{I}$ of dimensions (n, n). The relevant matrices are shown in Appendix B

As before, $\mathbf{N}$ is rank deficient and we add a single constraint equation that the sum of the rankings equal zero that leads to the solution for vector $\mathbf{x}$ (ratings) as

$$\mathbf{x} = \begin{Bmatrix} r_1 = 0.728 & r_7 = 4.750 & r_{13} = -1.477 \\ r_2 = 1.607 & r_8 = -3.161 & r_{14} = 3.814 \\ r_3 = 7.394 & r_9 = -6.455 & r_{15} = 4.447 \\ r_4 = 1.108 & r_{10} = -6.223 & r_{16} = 1.925 \\ r_5 = -2.670 & r_{11} = 0.439 & r_{17} = -4.006 \\ r_6 = 0.750 & r_{12} = -7.971 & r_{18} = 5.001 \end{Bmatrix} \tag{15}$$

The sum of the ratings in $\mathbf{x}$ is zero as per the constraint equation and the largest rating is team 3 with $r_3 = 7.394$ and the lowest rating is team 12 with $r_{12} = -7.971$.

As before we want the highest rating team to have a rating of 100, therefore 92.170 must be added to each value in the set (15) giving the ratings as

$$\text{ratings} = \begin{Bmatrix} r_1 = 93.334 & r_7 = 97.356 & r_{13} = 91.129 \\ r_2 = 94.213 & r_8 = 89.445 & r_{14} = 96.420 \\ r_3 = 100.000 & r_9 = 86.151 & r_{15} = 97.053 \\ r_4 = 93.714 & r_{10} = 86.383 & r_{16} = 94.531 \\ r_5 = 89.936 & r_{11} = 93.045 & r_{17} = 88.600 \\ r_6 = 93.356 & r_{12} = 84.635 & r_{18} = 97.607 \end{Bmatrix}$$

A Table of teams and least squares ratings is shown below ranked from largest to smallest ratings

Rank	Team	Rating	Rank	Team	Rating	Rank	Team	Rating
1	**3**(7)	100	7	**2**(4)	94.213	13	**5**(1)	89.936
2	**18**(5)	97.607	8	**4**(3)	93.714	14	**8**(1)	89.445
3	**7**(5)	97.356	9	**6**(2)	93.356	15	**17**(3)	88.600
4	**15**(3)	97.053	10	**1**(3)	93.334	16	**10**(0)	86.383
5	**14**(4)	96.420	11	**11**(4)	93.045	17	**9**(1)	86.151
6	**16**(4)	94.531	12	**13**(1)	91.129	18	**12**(1)	84.635

Table 10. Ranking of Least Squares Ratings from Qualifying, Principale, Complémentaire. (games won in parentheses)

It is interesting to note that the top two rated teams, 3 and 18, won the Principale and Complémentaire respectively, but if they were ranked according to their place in the two finals series, team 3 would be the top-ranked team having won the Principale and team 18 would be the 9th ranked team having won the Complémentaire.

Option 3: Ratings from Qualifying + Principale + Complémentaire with variable weights

We can use the observation equations for the 52 matches shown above (Qualifying + Principale + Complémentaire) but with variable weights. In the previous Least Squares cases we have treated all the matches as having the same importance and given each observation equation a weight of 1, that is $w = 1$. But it is possible give some matches more importance by giving them a greater weight. Suppose that Qualifying matches are given a weight $w = 1$, matches in the Complémentaire a weight $w = 2$ and matches in the Principale a weight $w = 3$. This weighting scheme will lead to a different set of normal equations $\mathbf{Nx} = \mathbf{t}$ and matrices $\mathbf{N}$ and $\mathbf{t}$ are shown in Appendix B.

As before, $\mathbf{N}$ is rank deficient and we add a single constraint equation that the sum of the rankings equal zero that leads to the solution for vector $\mathbf{x}$ (ratings) as

$$\mathbf{x} = \left\{ \begin{array}{lll} r_1 = 1.159 & r_7 = 4.641 & r_{13} = -1.573 \\ r_2 = 1.413 & r_8 = -4.149 & r_{14} = 4.142 \\ r_3 = 7.964 & r_9 = -6.664 & r_{15} = 3.635 \\ r_4 = 0.421 & r_{10} = -5.726 & r_{16} = 0.297 \\ r_5 = -2.807 & r_{11} = 1.636 & r_{17} = -5.043 \\ r_6 = 2.125 & r_{12} = -7.979 & r_{18} = 6.326 \end{array} \right\} \tag{16}$$

As before, giving the highest rated team a rating of 100 gives the ratings as

$$\text{ratings} = \left\{ \begin{array}{lll} r_1 = 93.195 & r_7 = 96.677 & r_{13} = 90.463 \\ r_2 = 93.449 & r_8 = 87.887 & r_{14} = 96.178 \\ r_3 = 100.000 & r_9 = 85.372 & r_{15} = 95.671 \\ r_4 = 92.457 & r_{10} = 86.310 & r_{16} = 92.333 \\ r_5 = 89.229 & r_{11} = 93.672 & r_{17} = 86.993 \\ r_6 = 94.161 & r_{12} = 84.239 & r_{18} = 98.362 \end{array} \right\}$$

A Table of teams and least squares ratings is shown below ranked from largest to smallest ratings

Rank	**Team**	**Rating**	**Rank**	**Team**	**Rating**	**Rank**	**Team**	**Rating**
1	**3**(7)	100	7	**11**(4)	93.672	13	**5**(1)	89.229
2	**18**(5)	98.362	8	**2**(4)	93.449	14	**8**(1)	87.887
3	**7**(5)	96.677	9	**1**(3)	93.195	15	**17**(3)	86.993
4	**14**(4)	96.178	10	**4**(3)	92.457	16	**10**(0)	86.310
5	**15**(3)	95.671	11	**16**(4)	92.333	17	**9**(1)	85.372
6	**6**(2)	94.161	12	**13**(1)	90.463	18	**12**(1)	84.239

Table 11. Ranking of Least Squares Ratings from Qualifying, Principale, Complémentaire with variable weights (games won in parentheses).

Note that the top six ranked teams from the Least Squares solution with equal weights (see Table 10) are 3, 18, 7, 15, 14 and 16. Team 6, with only two wins for the tournament, would appear to be an anomaly; ranked 6 with variable weights and 9 with equal weights. This is probably due to finishing in 8th place in the Qualifying with a small delta (-8) and a 2-point loss in the quarter-finals of the Principale.

Player Ratings using the Elo System

The Elo Rating System (Elo 1978) is a mathematical process based on a statistical model relating match results to underlying variables representing the abilities of a team or player; where player and team are interchangeable terms assuming a team rating is the average of the player's ratings in the team. The name "Elo" derives from Arpad Elo, the inventor of a system for rating chess players and his system, in various modified forms, is used for player or team ratings in many sports.

The Elo Rating System calculates, for every player or team, a numerical rating based on performance in competitions. A rating is a number (usually an integer) between 0 and 3000 that changes over time depending on the outcome of tournament games. The system depends on a curve defined by a *logistic function* (Langville & Meyer 2012, Glickman & Jones, 1999)

$$P_A = \frac{1}{1+10^{-\left(\frac{r_A - r_B}{b}\right)}} \tag{17}$$

where P_A is the probability of player A winning in a match A versus B given the player ratings $r_A, r_B > 0$ and shape parameter $b > 0$. Note $0 \le P_A \le 1$ and $P_B = 1 - P_A$ is the probability of B winning.

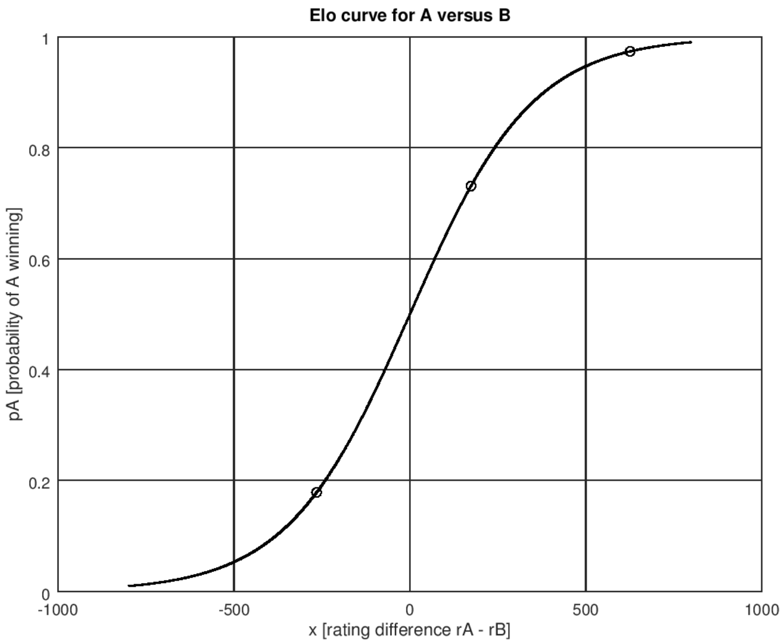

Figure 3. Elo curve: $P_A = \frac{1}{1+10^{-\left(\frac{x}{400}\right)}}$. P_A is the probability of A winning, $x = r_A - r_B$ is the rating difference and the shape parameter $b = 400$. The three points on the curve shown thus ○ have rating differences -265, 174 and 626 that correspond with probabilities 0.18, 0.73 and 0.97 respectively.

Additional information on the Logistic function can be found in Appendix C.

The curve in Figure 3 has the shape parameter $b = 400$ and this value is chosen so that a player rating difference of approximately 200 corresponds to a probability of winning of approximately 0.75.

For example, suppose two players A and B with ratings 1862 and 1671 respectively play a match. With the rating difference $r_A - r_B = 191$ and shape parameter $b = 400$, the probability of A winning is given by (17) as

$$P_A = \frac{1}{1+10^{-\left(\frac{1862-1671}{400}\right)}} = \frac{1}{1+10^{-\left(\frac{191}{400}\right)}} = \frac{1}{1+10^{-0.4775}} = 0.750$$

We might express this probability of A winning as:

(i) If A played B in 100 matches then A would win 75 of them, or

(ii) A has a 75% chance of winning.

With $x = r_A - r_B$ and $b = 400$ (17) becomes

$$P_A = \frac{1}{1+10^{-\left(\frac{x}{400}\right)}} \tag{18}$$

and this equation can be rearranged as $10^{\left(\frac{x}{400}\right)} = \frac{P_A}{1-P_A} = \frac{P_A}{P_B}$. Now using the rule for logarithms that if $d = \log_a N$ then $N = a^d$ and the expression for rating difference x is

$$x = 400\log_{10}\left(\frac{P_A}{1-P_A}\right) = 400\log_{10}\left(\frac{P_A}{P_B}\right) \tag{19}$$

Using (19) we can construct Table 12; a table of logistic probabilities P_A and $P_B = 1 - P_A$ corresponding to rating differences $x = r_A - r_B$.

For example, assuming that any value of P_A such that $0.745000 \le P_A \le 0.754999$ will be rounded to 0.75, the lower and upper bounds of the inequality, that is, $P_A = 0.745$ and $P_A = 0.754999$, give rating differences $x = 187$ and $x = 196$ respectively.

Rating difference	P_A	P_B	Rating difference	P_A	P_B	Rating difference	P_A	P_B
0–3	0.50	0.50	120–127	0.67	0.33	283–295	0.84	0.16
4–10	0.51	0.49	128–135	0.68	0.32	296–308	0.85	0.15
11–17	0.52	0.48	136–143	0.69	0.31	309–323	0.86	0.14
18–24	0.53	0.47	144–151	0.70	0.30	324–338	0.87	0.13
25–31	0.54	0.46	152–160	0.71	0.29	339–354	0.88	0.12
32–38	0.55	0.45	161–168	0.72	0.28	355–372	0.89	0.11
39–45	0.56	0.44	169–177	0.73	0.27	373–392	0.90	0.10
46–53	0.57	0.43	178–186	0.74	0.26	393–413	0.91	0.09
54–60	0.58	0.42	**187–196**	**0.75**	**0.25**	414–436	0.92	0.08
61–67	0.59	0.41	197–205	0.76	0.24	437–463	0.93	0.07
68–74	0.60	0.40	206–215	0.77	0.23	464–494	0.94	0.06
75–81	0.61	0.39	216–225	0.78	0.22	495–531	0.95	0.05
82–89	0.62	0.38	226–235	0.79	0.21	532–576	0.96	0.04
90–96	0.63	0.37	236–246	0.80	0.20	577–636	0.97	0.03
97–104	0.64	0.36	247–258	0.81	0.19	637–727	0.98	0.02
105–111	0.65	0.35	259–269	0.82	0.18	728–920	0.99	0.01
112–119	0.66	0.34	270–282	0.83	0.17	Over 920	1.00	0.00

Table 12. Logistic Probabilities for rating difference ranges for matches A v B where A is the higher rating team and shape parameter $b = 400$ (Elo 1978)

Updating Player Ratings

After a match A v B the player ratings are updated using

$$r_A = r'_A + K\left(W_A - P_A\right) \quad \text{and} \quad r_B = r'_B + K\left(W_B - P_B\right) \tag{20}$$

where r'_A, r'_B are the 'old' ratings for players A and B and r_A, r_B are the 'new' or updated ratings.

K (or the K-*Factor*) is a multiplier, $W_A = \left\{\begin{matrix} 1 \text{ if } A \text{ wins} \\ 0 \text{ if } A \text{ loses} \end{matrix}\right\}$, $W_B = \left\{\begin{matrix} 1 \text{ if } B \text{ wins} \\ 0 \text{ if } B \text{ loses} \end{matrix}\right\}$ and $P_A, P_B = 1 - P_A$ are probabilities of A and B winning and are determined prior to the match using (17) and the old ratings.

The ratings update formula (20) rewards a weaker player for defeating a stronger player to a greater degree than it rewards a stronger player for beating a weak opponent (Langville & Meyer 2012).

For example, if A is a strong player with rating $r_A = 1862$ and B is a weaker player with rating $r_B = 1671$ then from (17) with shape parameter $b = 400$

$$\begin{aligned} P_A &= \frac{1}{1 + 10^{-\left(\frac{1862-1671}{400}\right)}} \approx \frac{1}{1 + \frac{1}{3}} = \frac{3}{4} \\ P_B &= \frac{1}{1 + 10^{-\left(\frac{1671-1862}{400}\right)}} \approx \frac{1}{1 + 4} = \frac{1}{4} \end{aligned}$$

Therefore, the reward to the weaker player B for beating the stronger player A is

$$r_B - r'_B = K\left(1 - \frac{1}{4}\right) = \frac{3}{4}K$$

whereas if the stronger player defeats the weaker player, then the reward is only

$$r_A - r'_A = K\left(1 - \frac{3}{4}\right) = \frac{1}{4}K$$

It should be noted that if the change in player ratings are denoted $dr_A = r_A - r'_A$ and $dr_B = r_B - r'_B$ then from (20)

$$\begin{aligned} dr_A + dr_B &= K\left(W_A - P_A\right) + K\left(W_B - P_B\right) \\ &= K\left(W_A + W_B\right) - K\left(P_A + P_B\right) \\ &= 0 \end{aligned} \tag{21}$$

This means that the sum of player ratings before a tournament of matches begins will be the same for the updated player ratings at the conclusion of the tournament.

The shape parameter b

The shape parameter b in (17) is related to the spread of the ratings and the gradient of the Elo curve at the midpoint, and as b decreases in value the curve becomes steeper at the mid-point. See Figure 4 below where the solid curve has $b = 400$ and the dashed curve has $b = 600$

To examine the relationship between b and the rating difference $r_A - r_B$ we may, with a little bit of algebra and the fact that $P_B = 1 - P_A$, find that (17) can be rearranged to give

$$P_A = P_B 10^{\left(\frac{r_A - r_B}{b}\right)} \tag{22}$$

If the rating difference $r_A - r_B = 0$ then $P_A = P_B 10^0 = P_B$ and both players have an equal probability of winning.

If the rating difference $r_A - r_B = b$ then $P_A = P_B 10^1 = 10P_B$ and player A has a probability of winning that is 10 times player B*'s* probability.

If the rating difference $r_A - r_B = \frac{1}{2}b$ then $P_A = P_B 10^{\frac{1}{2}} = \sqrt{10}P_B \approx 3P_B$ and player A has a probability of winning that is approximately 3 times player B*'s* probability.

If the rating difference $r_A - r_B = \frac{1}{3}b$ then $P_A = P_B 10^{\frac{1}{3}} \approx 2P_B$ and player A has a probability of winning that is approximately twice player B*'s* probability.

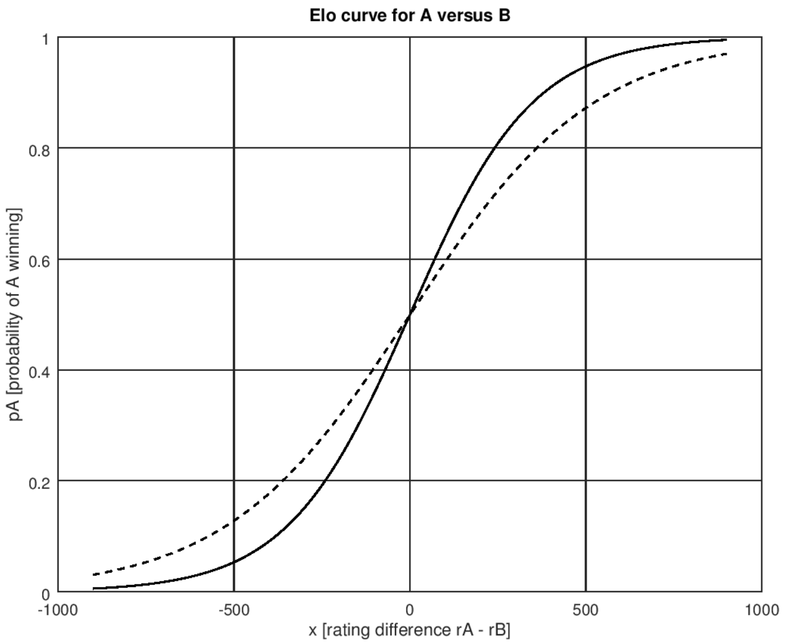

Figure 4. Elo curves: $P_A = \dfrac{1}{1 + 10^{-\left(\frac{x}{b}\right)}}$. P_A is the probability of A winning, $x = r_A - r_B$ is the rating difference and the shape parameter $b = 400$ gives the solid curve and $b = 600$ gives the dashed curve.

A value of $b = 400$ for ratings ranging between 500 and 2500 (rating difference ± 1000) would seem to be reasonable and for a rating difference of $r_A - r_B = 200$ would accord player A a probability of 0.75 and B a probability of 0.25, that is, player A is 3 times more likely to win than player B. See examples above where the rating difference is $\frac{1}{2}b$ and Table 12.

Choosing an appropriate shape parameter b is one of the ways of 'fine tuning' the Elo System for ratings in a particular sport. The value $b = 400$ is from the world of chess (Langville & Meyer 2012) but it is also used in other sports, for example, in Australian Football League (AFL) ratings (Elo Predicts) and World Football (World Football Elo Ratings).

The K-Factor

The 'K-factor' as it is known in chess circles is the multiplier K in the rating update formula (20). Various values of K are used; in chess $K = 32$ is common but can vary depending on the type of tournament and the experience of the player (Langville & Meyer 2012). If K is a large number, say $K > 50$ then player ratings will have large fluctuations and if K is small, say $K < 10$ then player ratings will change by only small amounts.

The value of K has a loose connection with the range of ratings and the number of wins required to progress through different levels of expertise. Suppose that two players A and B have a true rating difference of 40 points with A the better player. But, they are assigned ratings $r_A = 1100$ and $r_B = 2600$ by their sport association whose rating formula use $b = 400$ in (17) and $K = 32$ in (20). They then engage in a series of matches. A wins the first match, even though the assigned ratings have *A's* chance of success at 0.02% and his assigned rating increases by 32 to 1132. *B's* assigned rating decreases by 32 to 2568. A continues to win against *B,* since they are actually the better player, and at the beginning of the 28th match their ratings are $r_A = 1869$ and $r_B = 1831$ – a difference of 38 points, that is close to the supposed true rating difference. This indicates that approximately 30 matches might be required to establish meaningful ratings.

Some ranking systems use staggered K-factors. For example, the United States Chess Federation (USCF) has a staggered K-factor according to three main rating ranges of:

- $K = 32$, for players below 2100.
- $K = 24$, for players between 2100 and 2400.
- $K = 16$, for players above 2400.

The World Chess Federation (FIDE) also has a staggered K-factor

- $K = 40$, for players new to the rating list until the completion of events with a total of 30 games and for all players until their 18th birthday, as long as their rating remains under 2300.
- $K = 20$, for players with a rating always under 2400.
- $K = 10$, for players with any published rating of at least 2400 and at least 30 games played in previous events. Thereafter it remains permanently at 10.

The choice of an appropriate K-factor (or staggered K-factors) is dependent on the sport and requires some analysis of historical data. Simulations of matches with different K-factors may also be useful in determining appropriate values.

Elo Ratings in a Petanque Tournament.

Example 8

To show how the Elo Rating System might be employed in a petanque tournament we will use results from the Qualifying rounds of the Dove Open Doubles 01-May-2016 (see Example 7) and the following team ratings

Team	Rating	Team	Rating	Team	Rating
1	1688	**7**	1862	**13**	1621
2	1709	**8**	1651	**14**	1746
3	1984	**9**	1290	**15**	**2000**
4	1681	**10**	1273	**16**	1916
5	1488	**11**	1460	**17**	1665
6	1719	**12**	**1200**	**18**	1707

Table 13. Team Ratings prior to start of Dove Open Doubles

[The ratings in Table 13 are derived from the ratings obtained by Least Squares (see Example 7, Option 1, Table 9) that have been transformed to integer values between 2000 (highest rated team) and 1200 (lowest rated team)]

	Team A		Team B						Rating changes		Updated ratings	
	No.	r'_A	No.	r'_B	W_A	P_A	W_B	P_B	dr_A	dr_B	r_A	r_B
R1	**16**	1916	**9**	1290	1	0.973	0	0.027	0.848	-0.848	1917	1289
	18	1707	**14**	1746	0	0.444	1	0.556	-14.211	14.211	1693	1760
	5	1488	**11**	1460	1	0.540	0	0.460	14.713	-14.713	1503	1445
	7	1862	**1**	1688	1	0.731	0	0.269	8.596	-8.596	1871	1679
	2	1709	**12**	1200	1	0.949	0	0.051	1.622	-1.622	1711	1198
	8	1651	**4**	1681	1	0.457	0	0.543	17.378	-17.378	1668	1664
	15	2000	**17**	1665	1	0.873	0	0.127	4.062	-4.062	2004	1661
	13	1621	**6**	1719	0	0.363	1	0.637	-11.603	11.603	1609	1731
	3	1984	**10**	1273	1	0.984	0	0.016	0.525	-0.525	1985	1272
R2	**2**	1711	**14**	1760	0	0.430	1	0.570	-13.758	13.758	1697	1774
	7	1871	**15**	2004	0	0.317	1	0.683	-10.158	10.158	1861	2014
	8	1668	**16**	1917	0	0.193	1	0.807	-6.162	6.162	1662	1923
	5	1503	**17**	1661	0	0.287	1	0.713	-9.187	9.187	1494	1670
	3	1985	**6**	1731	1	0.812	0	0.188	6.020	-6.020	1991	1725
	4	1664	**11**	1445	1	0.779	0	0.221	7.068	-7.068	1671	1438
	9	1289	**18**	1693	0	0.089	1	0.911	-2.849	2.849	1286	1696
	13	1609	**12**	1198	1	0.914	0	0.086	2.746	-2.746	1612	1195
	10	1272	**1**	1679	0	0.088	1	0.912	-2.804	2.804	1269	1682
R3	**3**	1991	**14**	1774	1	0.777	0	0.223	7.131	-7.131	1998	1767
	15	2014	**16**	1923	0	0.628	1	0.372	-20.097	20.097	1994	1943
	17	1670	**6**	1725	0	0.422	1	0.578	-13.488	13.488	1657	1738
	5	1494	**18**	1696	0	0.238	1	0.762	-7.621	7.621	1486	1704
	4	1671	**13**	1612	1	0.584	0	0.416	13.309	-13.309	1684	1599
	8	1662	**1**	1682	0	0.471	1	0.529	-15.080	15.080	1647	1697
	2	1697	**7**	1861	0	0.280	1	0.720	-8.963	8.963	1688	1870
	11	1438	**10**	1269	1	0.726	0	0.274	8.778	-8.778	1447	1260
	9	1286	**12**	1195	0	0.628	1	0.372	-20.097	20.097	1266	1215
R4	**3**	1998	**16**	1943	1	0.578	0	0.422	13.488	-13.488	2011	1930
	18	1704	**7**	1870	0	0.278	1	0.722	-8.888	8.888	1695	1879
	14	1767	**4**	1684	0	0.617	1	0.383	-19.751	19.751	1747	1704
	15	1994	**6**	1738	1	0.814	0	0.186	5.964	-5.964	2000	1732
	1	1697	**13**	1599	1	0.637	0	0.363	11.603	-11.603	1709	1587
	5	1486	**2**	1688	0	0.238	1	0.762	-7.621	7.621	1478	1696
	17	1657	**8**	1647	1	0.514	0	0.486	15.540	-15.540	1673	1631
	11	1447	**12**	1215	1	0.792	0	0.208	6.664	-6.664	1454	1208
	10	1260	**9**	1266	0	0.491	1	0.509	-15.724	15.724	1244	1282

Table 14. Elo rating updates for Qualifying rounds of Dove Open Doubles

Table 14 shows team ratings (initial r'_A, r'_B and updated r_A, r_B) for the 4 rounds of Qualifying matches (18 players). For each match, the first team is Team A and the second team is Team B. The probability P_A is calculated using (17) with shape parameter $b = 400$ and initial ratings r'_A, r'_B, and $P_B = 1 - P_A$.

$W_A = \begin{Bmatrix} 1 \text{ if } A \text{ wins} \\ 0 \text{ if } A \text{ loses} \end{Bmatrix}$, $W_B = \begin{Bmatrix} 1 \text{ if } B \text{ wins} \\ 0 \text{ if } B \text{ loses} \end{Bmatrix}$ and dr_A, dr_B are rating changes. The updated ratings r_A, r_B are calculated using (20) with $K = 32$.

As an example, consider the 2nd match of Round 3, Team 15 v Team 16. Team A is the first named team (15) and Team B is the second named team (16). The initial ratings are $r'_A = 2014$, $r'_B = 1923$ and Team A loses so $W_A = 0$ and $W_B = 1$. There are three steps to updating Elo ratings.

First, using (17) with shape parameter $b = 400$ the probabilities P_A, P_B are

$$P_A = \frac{1}{1+10^{-\left(\frac{r'_A - r'_B}{400}\right)}} = \frac{1}{1+10^{-\left(\frac{2014-1923}{400}\right)}} = \frac{1}{1+10^{-\left(\frac{91}{400}\right)}} = \frac{1}{1+0.592} = 0.628$$

$$P_B = 1 - P_A = 0.372$$

Second, using (20) with $K = 32$ the rating updates dr_A, dr_B are

$$\begin{aligned} dr_A &= r_A - r'_A = K\left(W_A - P_A\right) = 32\left(0 - 0.628\right) = -20.097 \\ dr_B &= r_B - r'_B = K\left(W_B - P_B\right) = 32\left(1 - 0.372\right) = +20.097 = -dr_A \end{aligned}$$

Lastly, the updated ratings r_A, r_B are

$$\begin{aligned} r_A &= r'_A + dr_A = 2014 - 20.097 = 1993.903 = 1994 \text{ (nearest integer)} \\ r_B &= r'_B + dr_B = 1923 + 20.097 = 1943.097 = 1943 \text{ (nearest integer)} \end{aligned}$$

Table 15 shows the updated team ratings at the end of the Qualifying rounds.

Team	Rating	Team	Rating	Team	Rating
1	1709 (3,+21)	**7**	1879 (3,+17)	**13**	1587 (1,-34)
2	1696 (2,-13)	**8**	1631 (1,-20)	**14**	1747 (2,+1)
3	2011 (4,+27)	**9**	1282 (1,-8)	**15**	**2000** (3, 0)
4	1704 (3,+23)	**10**	1244 (0,-29)	**16**	1930 (3,+14)
5	1478 (1,-10)	**11**	14454 (2,-6)	**17**	1673 (2,+8)
6	1732 (2,+13)	**12**	**1208** (1,+8)	**18**	1695 (2,-12)

Table 15. Team Ratings after the Qualifying rounds of Dove Open Doubles. Values in parentheses are games won and increase/decrease in ratings from starting values (see Table 13). Ratings in bold are highest/lowest

Note that the sum of the initial ratings in Table 13 is 29660 and after the four Qualifying rounds, the sum of the updated ratings in Table 15 is 29660, as we should expect according to (21).

Incorporating Delta into Elo ratings

In petanque, delta $\left(\delta\right)$ is points difference and is defined as

$$\delta = \text{ points-for} - \text{ points-against}$$

and since there can be no draws in a petanque match $1 \le \left|\delta\right| \le 13$ where $\left|\delta\right|$ means the absolute value of δ, that is, $\left|\delta\right|$ is the non-negative value of δ without regard to its sign.

It may be desirable to incorporate delta in the Elo rating formula and this could be achieved by a *Margin of Victory Multiplier* (Silver 2014) that we denote by M

$$M = \ln\left(\left|\delta\right| + 1\right)\frac{2.2}{\frac{r_W - r_L}{1000} + 2.2} \tag{23}$$

where $\ln\left(x\right)$ denotes the natural logarithm of x and r_W, r_L are Elo ratings of winning (W) and losing (L) teams respectively.

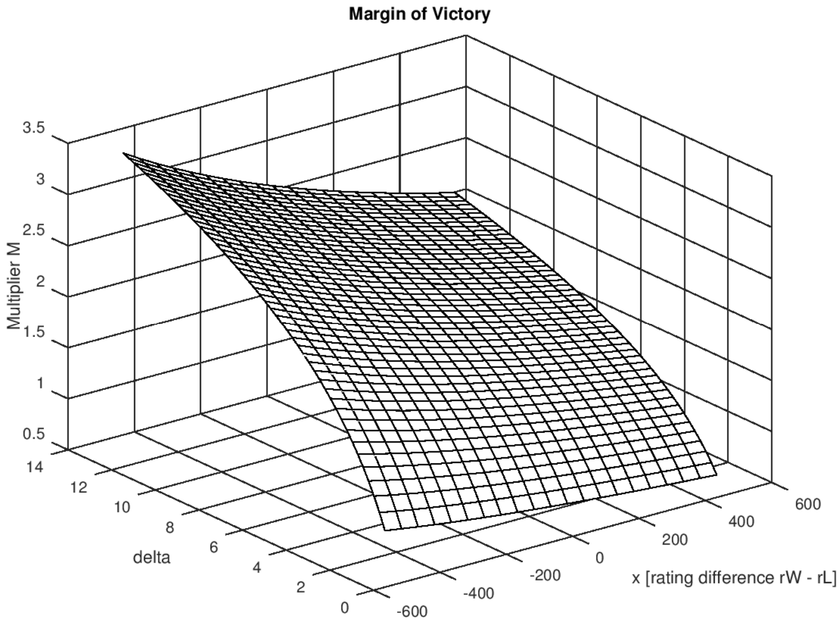

Figure 5. Margin of Victory Multiplier M using (23) with $1 \le \delta \le 13$ and $-500 \le \left(r_W - r_L\right) \le 500$

The multiplier M could be inserted into the rating update formula (20) to give

$$r_A = r_A' + KM\left(W_A - P_A\right) \quad \text{and} \quad r_B = r_B' + KM\left(W_B - P_B\right) \tag{24}$$

Equation (23) was developed for American Football (NFL) and it rewards underdogs who win and discounts favourites who have big wins.

For example, in Round 1 of the Qualifying rounds in the Dove Open Doubles (see Tables 4 and 14) team 2 scores 13 and team 12 scores 0 for $\delta = 13$ (the maximum value). $r_W = 1709$ and $r_L = 1200$, and (23) gives the Margin of Victory Multiplier M as

$$M = \ln\left(\left|\delta\right| + 1\right)\frac{2.2}{\dfrac{r_W - r_L}{1000} + 2.2} = \ln\left(14\right)\frac{2.2}{\dfrac{1709 - 1200}{1000} + 2.2} = 2.639\frac{2.2}{2.709} = 2.143$$

and the rating changes would have been

$$\begin{aligned} dr_A &= r_A - r_A' = KM\left(W_A - P_A\right) = 32 \times 2.143\left(0 - 0.628\right) = -43.066 \\ dr_B &= r_B - r_B' = KM\left(W_B - P_B\right) = 32 \times 2.143\left(1 - 0.372\right) = +43.066 \end{aligned}$$

rather than ± 20.097

If the situation had have been reversed and team 12 had beaten team 2 13 points to zero (an underdog victory) then

$$M = \ln\left(\left|\delta\right| + 1\right)\frac{2.2}{\dfrac{r_W - r_L}{1000} + 2.2} = \ln\left(14\right)\frac{2.2}{\dfrac{1200 - 1709}{1000} + 2.2} = 2.639\frac{2.2}{1.691} = 3.433$$

and the rating changes would have been ± 68.990 rather than ± 20.097.

Other forms of Margin of Victory Multipliers could be developed and alternatives are given in the literature, for example, Langville & Meyer (2012). This could be a topic for further investigation.

Conclusion

We have given a detailed description of three methods of rating players or teams in petanque: (i) the VPCI Player Rating System, (ii) using Least Squares and (iii) using the Elo system and examples each method are given.

The VPCI Player Rating System is based on a simple formula that awards tournament-points t to players according to their ranking position at the conclusion of a tournament and each player in a team receives the same tournament points. For example, in a club-hosted tournament (the most common type) the team or player finishing first has a rank $R = 1$ and receives $t = 1000$ tournament-points, teams finishing 2nd, 3rd, 4th, etc. receive $t = 750, 600, 500$ tournament-points respectively. A player's rating is then either the sum of their tournament-points divided by 10 if they have played in 10 or fewer tournaments in the preceding 12-month period, or the average of their 10 best results in the preceding 12-month period. The top-rated players in Victoria have ratings between 900 and 1300.

Determining player ratings using Least Squares is more complicated and is based on a set of observation equations having the form $v + r_W - r_L = s_W - s_L$ where r_W, r_L are the unknown ratings of the winning (W) and losing (L) teams and s_W, s_L are the game scores. v is a residual (a small unknown correction) and the least squares process yields ratings that minimize the sum of the squares of the weighted residuals. We have shown an example of ratings determined by least squares for the four Qualifying rounds of a triples tournament where 18 teams competed. There were $n = 36$ equations in $u = 18$ unknown ratings which required the use of sophisticated software for the solution. The least squares process also allows for differing weights of observation equations. For example, equations related to Qualifying matches could have weight $w = 1$, equations relating to the Principale and Complémentaire could have weights $w = 3$ and $w = 2$ respectively. This weighting scheme is arbitrary but could be connected with some prior knowledge of team strengths.

The Elo System assumes a team's performance in tournaments accords with a bell-shaped probability density function. That is, large wins and crushing defeats are relatively rare and a high proportion of results fall within a reasonable range of delta. The cumulative distribution function, that is the integral of the density function, is assumed to be a Logistic function whose curve is a symmetric S-shape, and it is from this curve, and its equation, that we are able to determine P_A that is the probability of success for team A in a match A versus B with $P_B = 1 - P_A$ and team ratings r_A, r_B. The shape of the adopted curve is chosen so that a rating difference $r_A - r_B = 200$ gives the probability of team A winning of 0.75. These probabilities P_A, P_B are combined with actual win/loss results (win = 1, loss = 0) in a simple rating update formula that is weighted with a suitable factor K. The formula are relatively simple and probabilities can be evaluated on any scientific calculator that allows exponentiation, or from pre-computed tables if desired. The rating update formula has the attractive feature of rewarding weaker teams for defeating stronger teams to greater degree than it rewards stronger teams for defeating weaker opponents. There is also the option in this system of modifying the K-factor for matches between higher rated teams or matches in higher rated tournaments such as State or National tournaments. In addition, the K-factor could be modified by including a function of game scores so that rewards increase for higher delta victories.

Of the three rating systems we have investigated, the Least Squares ratings would appear to be the more difficult to use in practice and would require sophisticated software. The Elo system is attractive as it rewards teams who win against higher rated opponents no matter where they may finish at the end of a tournament. Indeed, in a tournament, a team may lose more matches than they win and still improve their rating. This might encourage greater participation in tournaments, as teams with little expectation of overall victory could have rating improvement as their goal.

The simplest of the three systems is the VPCI Player Rating System, and it is embedded within *Mypetanque* where a ranking of players is shown with their tournament history for the previous 12 months. This is a very useful resource for coaches and administrators.

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APPENDIX A: Earlier versions of VPCI Player rating System

The original VPCI Player Rating System (Wells 2016).

A player's rating r was calculated from base-points b, tournament-points t and averages of tournament points in the following 3-step sequence:

[1] Using tournament results submitted to *Mypetanque,* a simple formula was used to assign base-points b to players according to their final ranking R in a particular tournament, as: 100 base points for 1st $\left(R=1\right)$, 75 base points for 2nd, 60 base points for 3rd, and so on, down to 16⅔ base points for 16th position $\left(R=16\right)$. For all other players in the tournament a 'participation score' of 15 base-points was awarded. The formula for assigning base-points was

$$b = \frac{300}{R+2} \quad \text{for} \quad R \le 16 \quad \text{or} \quad b = 15 \quad \text{for} \quad R > 16$$

R	b	R	b	R	b	R	b
1	100	6	37.5	11	23.0769	16	16.6667
2	75	7	33.3333	12	21.4286	17	15
3	60	8	30	13	20	18	15
4	50	9	27.2727	14	18.75	:	:
5	42.8571	10	25	15	17.6471	N	15

Table A1. Base-points b for ranking position R in a tournament of N players for the superseded Original Player Rating System.

[2] Tournament-points t were obtained by multiplying base-points b by two factors c and d that depended on the tournament type and tournament class.

$$t = b \times c \times d$$

The tournament type was one of the set {Triples, Doubles, Singles} and the tournament class was one of the set {Regional, Club, State, National}. Here Club means a Club-hosted event open to all licensed players in Australia. The values of the factors of factors c and d are shown in Table A2.

Tournament type	factor c	Tournament class	factor d
Triples	1.00	Regional	0.80
Doubles	1.25	Club	1.00
Singles	1.50	State	1.28
		National	1.60

Table A2. Factors c and d for tournament type and class for the superseded Original Player Rating System.

[3] A player's rating r was then one of three values depending on the number of tournaments T they have played in:

(i) $T > 10$ (More than 10 tournaments). Rating r is the average of tournament-points t of their 10 best results in the previous 12-month period; or

(ii) $T < 5$ (Less than 5 tournaments). Rating r is the total tournament-points t divided by 5; or

(iii) $5 \le T \le 10$ ($T = 5$, 6, 7, 8, 9 or 10). Rating r is the average tournament-points t.

The 2017 Player Rating System (Wells 2017).

In this modified system the computation of base-points b and then their transformation to tournament-points t was replaced with a table of tournament-points t for a range of different tournament classes. There was also no distinction between tournament types (singles, doubles, triples, mixed, etc.) and all players in a team receive the same tournament-points. [The modifications to the original 2015 rating system were the result of consultations with the stronger players who felt that the tournament-points should not extend to the lower ranked players in a tournament, and that players who played more often should be rewarded to a greater extent. Also, the rating-points t are integers and this was thought to be a useful simplification.]

A player's rating r is calculated from tournament-points t and averages of tournament points in the following 2-step sequence:

[a] Using tournament results submitted to *Mypetanque,* tournament-points t according to their ranking R in a particular tournament are obtained from Table A3.

[b] A player's rating r is then one of three values depending on the number of tournaments T they have played in:

(i) $T > 10$ (More than 10 tournaments). Rating r is the average of tournament-points t of their 10 best results in the previous 12-month period; or

(ii) $T < 5$ (Less than 5 tournaments). Rating r is the total tournament-points t divided by 5; or

(iii) $5 \leq T \leq 10$ ($T = 5, 6, 7, 8, 9$ or 10). Rating r is the average tournament-points t.

<table>
<tr><th>Tournament Rank R</th><th colspan="2">Regional/Social Tournament</th><th colspan="2">Club-Hosted Tournament</th><th colspan="2">State Championship</th><th colspan="2">PFA/National Tournament</th></tr>
<tr><td>1</td><td colspan="2">50</td><td colspan="2">100</td><td colspan="2">150</td><td colspan="2">200</td></tr>
<tr><td>2</td><td colspan="2">37</td><td colspan="2">74</td><td colspan="2">111</td><td colspan="2">150</td></tr>
<tr><td>3</td><td colspan="2">30</td><td colspan="2">60</td><td colspan="2">90</td><td colspan="2">120</td></tr>
<tr><td>4</td><td colspan="2">25</td><td colspan="2">50</td><td colspan="2">75</td><td colspan="2">100</td></tr>
<tr><td>5</td><td>21</td><td rowspan="4">18 with no play-off</td><td>42</td><td rowspan="4">36 with no play-off</td><td>63</td><td rowspan="4">54 with no play-off</td><td>84</td><td rowspan="4">72 with no play-off</td></tr>
<tr><td>6</td><td>18</td><td>36</td><td>54</td><td>72</td></tr>
<tr><td>7</td><td>16</td><td>32</td><td>48</td><td>64</td></tr>
<tr><td>8</td><td>14</td><td>28</td><td>42</td><td>56</td></tr>
<tr><td>9</td><td>12</td><td rowspan="3">11 with no play-off</td><td>24</td><td rowspan="3">22 with no play-off</td><td colspan="2">36</td><td colspan="2">48</td></tr>
<tr><td>10</td><td>11</td><td>22</td><td colspan="2">33</td><td colspan="2">44</td></tr>
<tr><td>11</td><td>9</td><td>18</td><td colspan="2">27</td><td colspan="2">36</td></tr>
<tr><td>12</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">24</td><td colspan="2">32</td></tr>
<tr><td>13</td><td colspan="2">1</td><td colspan="2">1</td><td>22</td><td rowspan="3">18 with no play-off</td><td>28</td><td rowspan="4">24 with no play-off</td></tr>
<tr><td>14</td><td colspan="2">1</td><td colspan="2">1</td><td>18</td><td>24</td></tr>
<tr><td>15</td><td colspan="2">1</td><td colspan="2">1</td><td>15</td><td>20</td></tr>
<tr><td>16</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td>16</td></tr>
<tr><td>17</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td>12</td><td rowspan="3">8 with no play-off</td></tr>
<tr><td>18</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td>8</td></tr>
<tr><td>19</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td>4</td></tr>
<tr><td>20</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td></tr>
<tr><td>:</td><td colspan="2">:</td><td colspan="2">:</td><td colspan="2">:</td><td colspan="2">:</td></tr>
<tr><td>N</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td></tr>
</table>

Table A3. Tournament-points t for ranking R in various tournament classes for the superseded VPCI 2017 Player Rating System

APPENDIX B: Matrices for Least Squares Ratings

Option 2: Ratings from Qualifying + Principale + Complémentaire (n = 52 matches)

$\mathbf{v} + \mathbf{Bx} = \mathbf{f}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{36} \\ v_{37} \\ v_{38} \\ \vdots \\ v_{52} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \vdots & & & & & & & & & & & & & & & & & \vdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \\ r_{15} \\ r_{16} \\ r_{17} \\ r_{18} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 4 \\ \vdots \\ 2 \\ 2 \\ 1 \\ \vdots \\ 10 \end{bmatrix}$$

All matches (Qualifying, Principale, Complémentaire) are regarded as having the same importance and each match is independent of other matches, hence we may assign a weight $w = 1$ to each observation equation forming a diagonal weight matrix $\mathbf{W} = \mathbf{I}$ of dimensions $\left(n, n\right)$. This leads to the normal equations $\left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ or $\mathbf{N}\mathbf{x} = \mathbf{t}$ where

$$\mathbf{N} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 7 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & -1 & -1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 7 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 7 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 7 \end{bmatrix}$$

and the transpose of the vector of numeric terms

$$\mathbf{t}^T = \begin{bmatrix} 5 & 15 & 45 & 4 & -18 & -10 & 15 & -16 & -21 & -27 & 21 & -26 & -7 & -5 & 15 & -2 & -23 & 35 \end{bmatrix}$$

Option 3: Ratings from Qualifying + Principale + Complémentaire with variable weights

With Qualifying matches having a weight $w = 1$, matches in the Complémentaire a weight $w = 2$ and matches in the Principale a weight $w = 3$ the coefficient matrix **N** and the vector of numeric terms **t** are

$$\mathbf{N} = \begin{bmatrix}
7 & 0 & 0 & 0 & 0 & 0 & -4 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -2 \\
0 & 0 & 13 & 0 & 0 & -4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -4 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -3 & 0 & 0 \\
0 & -3 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -4 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
-4 & -1 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & -3 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 6 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -2 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & -2 & 0 & -1 & 10 & -1 & 0 & 0 & 0 & 0 & -2 & -2 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 6 & 0 & 0 & 0 & 0 & -2 \\
0 & -1 & -4 & -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & -3 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 7 & -1 & -1 & 0 \\
0 & 0 & -4 & -3 & 0 & 0 & -3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 13 & 0 & 0 \\
0 & -2 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & 0 & 0 & -1 & 0 & 10 & 0 \\
0 & -2 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -2 & 0 & -2 & -1 & 0 & 0 & 0 & 10
\end{bmatrix}$$

and

$$\mathbf{t}^T = \begin{bmatrix} 1 & 19 & 83 & 2 & -24 & -14 & 31 & -25 & -23 & -27 & 38 & -26 & -18 & -11 & 11 & -36 & -42 & 61 \end{bmatrix}$$

APPENDIX C: Logistic function

The Logistic function has the following general form

$$y = \frac{A_1 - A_2}{1 + e^{\left(\frac{x-a}{b}\right)}} + A_2 \tag{25}$$

The curve of the Logistic function – the Logistic curve – is shown in Figure C1 and is a sigmoid or S-shaped curve that is symmetric about a midpoint and approaches upper and lower asymptotes as x approaches $\pm\infty$

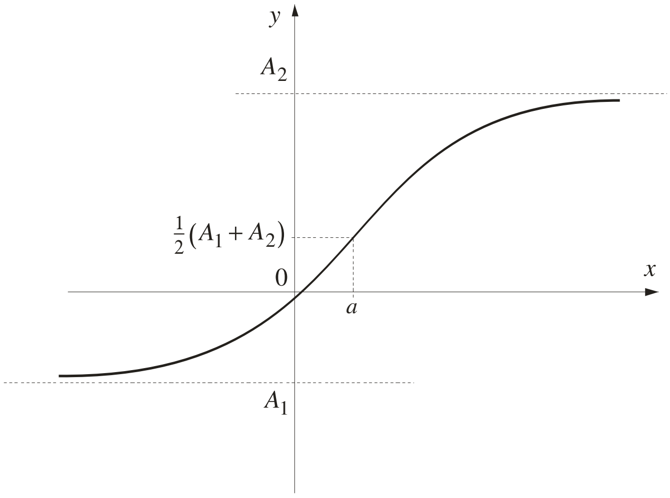

Figure C1. Logistic curve

$y = A_1$ is the lower asymptote of the curve when $x \to -\infty$ and $y = A_2$ is the upper asymptote of the curve when $x \to +\infty$. The curve is symmetric about the midpoint $\left[a, \frac{1}{2}\left(A_1 + A_2\right)\right]$ and the gradient of the curve at the midpoint is related to b and $b > 0$.

A more familiar form of the Logistic function is obtained when $A_1 = 0$ and $A_2 = 1$ giving

$$y = \frac{1}{1 + e^{-\left(\frac{x-a}{b}\right)}} \tag{26}$$

The curve of this function has the same form as the cumulative distribution curve of the Logistic distribution: $F_X\left(x\right) = \dfrac{1}{1 + e^{-\left(\frac{x-a}{b}\right)}} = \dfrac{1}{1 + e^{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)}}$ where $F_X\left(x\right) = P\left(X \le x\right)$, μ is the mean, σ is the positive square-root of the variance σ^2 and location and shape parameters a and b respectively are $a = \mu$ $b = \dfrac{\sigma\sqrt{3}}{\pi}$. $P\left(X \le x\right)$ means the probability of the random variable X being less than or equal to x..

Elo's logistic function is similar to (26) with 10 replacing e as a base, the location parameter $a = 0$ and $x = r_A - r_B$ is the rating difference between players A and B. [Note that $e = 2.7182818284\ldots$ is the base of natural logarithms].

For a detailed discussion of the Logistic function see Deakin (2018).